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A Generalization of the Terwilliger Algebra

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P. M. Terwilliger (1992, *J. Algebraic Combin.* **1**, 363–388) considered the $\mathbb{C}$-algebra generated by a given Bose Mesner algebra M and the associated dual Bose Mesner algebra M^*. This algebra is now known as the Terwilliger algebra and is usually denoted by T. Terwilliger showed that each vanishing intersection number and Krein parameter of M gives rise to a relation on certain generators of T. These relations determine much of the structure of T, thought not all of it in general. To illuminate the role these relations play, we consider a certain generalization $\mathcal{T}$ of T. To go from T to $\mathcal{T}$, we replace M and M^* with a pair of dual character algebras C and C^*. We define $\mathcal{T}$ by generators and relations; intuitively $\mathcal{T}$ is the associative $\mathbb{C}$-algebra with identity generated by C and C^* subject to the analogues of Terwilliger's relations. $\mathcal{T}$ is infinite dimensional and noncommutative in general. We construct an irreducible $\mathcal{T}$-module which we call the primary module; the dimension of this module is the same as that of C and C^*. We find two bases of the primary module; one diagonalizes C and the other diagonalizes C^*. We compute the action of the generators of $\mathcal{T}$ on these bases. We show $\mathcal{T}$ is a direct sum of two sided ideals $\mathcal{T}_0$ and $\mathcal{T}_1$ with $\mathcal{T}_0$ isomorphic to a full matrix algebra. We show that the irreducible module associated with $\mathcal{T}_0$ is isomorphic to the primary module. We compute the central primitive idempotent of $\mathcal{T}$ associated with $\mathcal{T}_0$ in terms of the generators of $\mathcal{T}$.

1. INTRODUCTION

There is an object in algebraic combinatorics known as a Bose Mesner algebra. There are several equivalent definitions [9, 17, 32], but one that is particularly compact is the following [17, 32]. Let n denote a positive integer, let $M_n(\mathbb{C})$ denote the $\mathbb{C}$-algebra of all n by n matrices with complex entries, and let $J \in M_n(\mathbb{C})$ denote the matrix whose entries are all 1. By a Bose Mesner algebra of order n we mean a commutative subalgebra M of $M_n(\mathbb{C})$ which contains J and which is closed under transposition

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and entrywise multiplication. The vector space M together with entrywise multiplication is a commutative $\mathbb{C}$-algebra with identity J; we refer to this algebra as M'. To avoid dealing directly with the entrywise product, it is convenient to consider a certain subalgebra M^* of $M_n(\mathbb{C})$ which is isomorphic to M'; this algebra is constructed as follows. For all $X \in M$, let $\rho(X)$ denote the diagonal matrix in $M_n(\mathbb{C})$ whose iith entry is equal to X_{i1}, for $1 \le i \le n$. For example, $\rho(J) = I$, the identity matrix in $M_n(\mathbb{C})$. Observe that the map ρ: $M \to M_n(\mathbb{C})$ is linear and let M^* denote the image of M under ρ. Since M is closed under entrywise multiplication and contains J, we see that M^* is closed under ordinary matrix multiplication and contains I. It follows that M^* is a subalgebra of $M_n(\mathbb{C})$, and one can show that ρ: $M' \to M^*$ is an isomorphism of $\mathbb{C}$-algebras [41]. The subalgebra T of $M_n(\mathbb{C})$ generated by M and M^* is known as the subconstituent algebra or the Terwilliger algebra [41]. It has been used to study P- and Q-polynomial association schemes [18, 41], group association schemes [8, 10], strongly regular graphs [44], Doob schemes [40], and association schemes over the Galois rings of characteristic four [31]. Other work involving the Terwilliger algebra can be found in [19–25, 27–29, 42, 43].

In this paper we introduce a generalization $\mathcal{T}$ of the Terwilliger algebra. We define $\mathcal{T}$ by generators and relations; in general, the result is infinite dimensional and noncommutative. Before describing $\mathcal{T}$, we set the stage by saying a bit more about M, M^*, and T.

The algebras M and M^* each have two bases of interest to us. To obtain one basis of M, observe that M' is semisimple, since it contains no nonzero nilpotent elements [37, Theorem 3.9]. Since M' is also commutative, it has a basis $A_0, \ldots, A_d$ consisting of mutually orthogonal idempotents. These matrices have all entries equal to zero or one and their sum is J. Moreover, for $0 \le i \le d$ there exists a positive integer k_i such that each row and column of A_i contains exactly k_i ones; this can be shown using the fact that A_i commutes with J. By definition of M we have $I \in M$ and it follows that I is one of $A_0, \ldots, A_d$; by convention we take $A_0 = I$. We define $E_i^* = \rho(A_i)$ for $0 \le i \le d$ and we observe that $E_0^*, \ldots, E_d^*$ is a basis of mutually orthogonal idempotents of M^*. To obtain the other basis of M, we show that M is semisimple. Observe that M is closed under complex conjugation, since it has a basis $A_0, \ldots, A_d$ whose entries are all real. By definition M is closed under transposition, so it is closed under the conjugate transpose. It follows that M is semisimple [26, p. 157]. Since M is also commutative, it has a basis $E_0, \ldots, E_d$ consisting of mutually orthogonal idempotents. The matrix $n^{-1}J$ is a rank one idempotent and so must be among $E_0, \ldots, E_d$; by convention we take $E_0 = n^{-1}J$. We define $A_i^* = n\rho(E_i)$ for $0 \le i \le d$. Observe that $A_0^*, \ldots, A_d^*$ is a basis for M^* and that $A_0^* = I$.

The inspiration for $\mathscr{T}$ is a result of Terwilliger concerning certain triple products in T; to describe this result, we recall two sets of parameters. Since $A_0, \ldots, A_d$ is a basis for M, there exist scalars p_{ij}^h such that

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \qquad (0 \le i, j \le d);$$

these are known as the intersection numbers of M. Similarly, there exist scalars p_{ij}^{h*} such that

$$A_i^* A_j^* = \sum_{h=0}^{d} p_{ij}^{h*} A_h^* \qquad (0 \le i, j \le d);$$

these are the known as the intersection numbers of M^* and also as the Krein parameters of M. Terwilliger showed in [41] that for $0 \le h, i, j \le d$ we have

$$E_h^* A_i E_j^* = 0 \qquad \text{iff } p_{ij}^h = 0$$

and

$$E_h A_i^* E_j = 0 \qquad \text{iff } p_{ij}^{h*} = 0.$$

We now describe the algebra $\mathscr{T}$. Let C denote an associative $\mathbb{C}$-algebra with a basis $x_0, \ldots, x_d$ such that

$$x_i x_j = \sum_{h=0}^{d} p_{ij}^h x_h \qquad (0 \le i, j \le d). \tag{1}$$

Observe that C is isomorphic to M; in fact, the linear map from M to C which maps A_i to x_i for $0 \le i \le d$ is an isomorphism of algebras. We write e_i to denote the image of E_i under this map and we observe that $e_0, \ldots, e_d$ is a basis of C consisting of mutually orthogonal idempotents. Similarly, let C^* denote an associative $\mathbb{C}$-algebra with a basis $x_0^*, \ldots, x_d^*$ such that

$$x_i^* x_j^* = \sum_{h=0}^{d} p_{ij}^{h*} x_h^* \qquad (0 \le i, j \le d). \tag{2}$$

Then C^* is isomorphic to M^* and the linear map from M^* to C^* which maps A_i^* to x_i^* for $0 \le i \le d$ is an isomorphism of algebras. We write e_i^* to denote the image of E_i^* under this map and we observe that $e_0^*, \ldots, e_d^*$ is a basis of C^* consisting of mutually orthogonal idempotents. We define $\mathscr{T}$ to be the associative $\mathbb{C}$-algebra with identity generated by $x_0, \ldots, x_d, x_0^*,$

$\ldots, x_d^*$ subject to the relations (1), (2), $x_0 = x_0^*$,

$$e_h^* x_i e_j^* = 0 \qquad \text{if } p_{ij}^h = 0 \qquad (0 \le h, i, j \le d) \tag{3}$$

and

$$e_h x_i^* e_j = 0 \qquad \text{if } p_{ij}^{h*} = 0 \qquad (0 \le h, i, j \le d). \tag{4}$$

The element $x_0 = x_0^*$ is the identity in $\mathscr{T}$. Intuitively, $\mathscr{T}$ is the associative $\mathbb{C}$-algebra with identity generated by C and C^* subject to the relations (3) and (4). We observe that T is a homomorphic image of $\mathscr{T}$.

In our description above, the algebra $\mathscr{T}$ is constructed from a given Bose Mesner algebra. However, in some sense we only needed the algebras C and C^*. These algebras are examples of character algebras; see Section 2 for a precise definition. In our main results we define $\mathscr{T}$ using character algebras; we do not assume an underlying Bose Mesner algebra.

We now describe our main results. We show that $x_0, \ldots, x_d$ remain linearly independent in $\mathscr{T}$, and hence form a basis for a subalgebra of $\mathscr{T}$ which is isomorphic to C. Similarly, we show $x_0^*, \ldots, x_d^*$ form a basis for a subalgebra of $\mathscr{T}$ which is isomorphic to C^*. For any $\mathscr{T}$-module V, we show the following are equivalent: (i) V is irreducible and $e_0 V \neq 0$, (ii) V is irreducible and $e_0^* V \neq 0$, (iii) $\dim e_i V = 1$ for $0 \le i \le d$, and (iv) $\dim e_i^* V = 1$ for $0 \le i \le d$. We show that there exists a $\mathscr{T}$-module which satisfies (i)–(iv) and that this module is unique up to isomorphism; we refer to this module as the primary module. Let V denote the primary module. We show that for every nonzero $u \in e_0 V$, the vectors $e_0^* u, \ldots, e_d^* u$ form a basis for V. Similarly, we show that for every nonzero $v \in e_0^* V$, the vectors $e_0 v, \ldots, e_d v$ form a basis for V. We compute the action of the elements x_i, x_i^*, e_i, and e_i^* on these bases. We consider certain two sided ideals $\mathscr{T}_0$ and $\mathscr{T}_1$ of $\mathscr{T}$. We show that $\mathscr{T}$ is the direct sum of $\mathscr{T}_0$ and $\mathscr{T}_1$ and that $\mathscr{T}_0$ is isomorphic to $M_{d+1}(\mathbb{C})$. We show that the irreducible $\mathscr{T}$-module associated with $\mathscr{T}_0$ is isomorphic to the primary module. We compute the central primitive idempotent of $\mathscr{T}$ associated with $\mathscr{T}_0$ in terms of the elements e_i and e_i^*.

We conclude this section by setting some notation. We write $\mathbb{C}$ to denote the field of complex numbers and $\mathbb{R}$ to denote the field of real numbers. For all $\alpha \in \mathbb{C}$, we write $\overline{\alpha}$ to denote the complex conjugate of α. From now on when we consider a matrix it will be convenient to index the rows and columns starting with zero. So for the rest of this paper we will regard matrices in $M_{d+1}(\mathbb{C})$ as having rows and columns indexed by $0, \ldots, d$. For $0 \le i, j \le d$ we write e_{ij} to denote the matrix in $M_{d+1}(\mathbb{C})$ with a 1 in the ijth entry and zeros in all other entries. Suppose A is a set and $f\colon A \to A$ is a map. We say f is an involution whenever f^2 is the identity map on A. In particular, the identity map on A is an involution.

2. CHARACTER ALGEBRAS

In this section we recall the notion of a character algebra (or C-algebra, for short) and state some basic results. For more information on character algebras, see [5, 9, 13, 30, 33, 35, 36]. We remark that in [35, 36] a character algebra is the same object as the double algebra of a finite Abelian classlike hypergroup.

DEFINITION 2.1. A *character algebra* $C = \langle X_0, \dots, X_d \rangle$ is a finite dimensional associative $\mathbb{C}$-algebra together with a basis $X_0, \dots, X_d$ having the following properties.

1. C is commutative.
2. X_0 is the multiplicative identity element of C.
3. Let p^h_{ij} $(0 \le h, i, j \le d)$ denote complex numbers such that

$$X_i X_j = \sum_{h=0}^{d} p^h_{ij} X_h \qquad (0 \le i, j \le d). \tag{5}$$

Then $p^h_{ij} \in \mathbb{R}$ for $0 \le h, i, j \le d$.

4. There exist a permutation $i \mapsto i'$ of $0, \dots, d$ and positive real numbers k_i $(0 \le i \le d)$ such that

$$p^0_{ij} = \delta_{ji'} k_i \qquad (0 \le i, j \le d). \tag{6}$$

5. The linear map $\tau: C \to C$ which satisfies $\tau(X_i) = X_{i'}$ for $0 \le i \le d$ is a $\mathbb{C}$-algebra isomorphism.

6. The linear map $\pi_0: C \to \mathbb{C}$ which satisfies $\pi_0(X_i) = k_i$ for $0 \le i \le d$ is a $\mathbb{C}$-algebra homomorphism.

We refer to the scalars k_i as the *valencies* of C. We refer to the scalars p^h_{ij} as the *structure constants* of C.

We observe that by (6) and commutativity, the permutation $'$ is unique and is an involution.

Remark. A character algebra whose structure constants are all nonnegative is essentially the same object as a table algebra. For more information on table algebras, see [1–4, 6, 7, 11, 12, 14–16, 45, 46].

We present four examples of character algebras.

EXAMPLE 2.2 [9, Sects. II.2, II.3]. Suppose M is a Bose Mesner algebra and $A_0, \dots, A_d$ are as in the Introduction. Then $M = \langle A_0, \dots, A_d \rangle$ is a character algebra. In this example, τ is the transpose map and π_0 is the character of M associated with E_0.

EXAMPLE 2.3 [9, Sects. II.2, II.3]. Let the Bose Mesner algebra M be as in the previous example, let M^* denote the dual Bose Mesner algebra associated with M, and define $A_0^*, \dots, A_d^*$ as in the Introduction. Then $M^* = \langle A_0^*, \dots, A_d^* \rangle$ is a character algebra. In this example, τ is the complex conjugation map and π_0 is the character of M^* associated with E_0^*.

EXAMPLE 2.4 [13, pp. 26–28]. Suppose G is a finite group and let $\mathbb{C}G$ denote the group algebra of G over $\mathbb{C}$. Let $C_0 = \{1_G\}, C_1, \dots, C_d$ denote the conjugacy classes of G and write

$$X_i = \sum_{x \in C_i} x \qquad (0 \le i \le d).$$

Then $C = \langle X_0, \dots, X_d \rangle$ is a character algebra, with $C_{i'} = \{x^{-1} \mid x \in C_i\}$ and $k_i = |C_i|$ for $0 \le i \le d$. We observe that C is the center of $\mathbb{C}G$.

EXAMPLE 2.5 [13, pp. 26–28]. Suppose G is a finite group. Let χ_0 denote the trivial complex character of G, and let $\chi_1, \dots, \chi_d$ denote the remaining irreducible complex characters of G. Write

$$X_i = d_i \chi_i \qquad (0 \le i \le d),$$

where $d_i = \deg \chi_i$ for $0 \le i \le d$. Then $C = \langle X_0, \dots, X_d \rangle$ is a character algebra, with $\chi_{i'} = \overline{\chi_i}$ and $k_i = d_i^2$ for $0 \le i \le d$.

The structure constants and valencies of a character algebra satisfy certain relations; we recall those which will be useful later on.

PROPOSITION 2.6 [9, pp. 88, 89]. *Suppose $C = \langle X_0, \dots, X_d \rangle$ is a character algebra. Then*

(i) $p_{ij}^h = p_{ji}^h \qquad (0 \le h, i, j \le d)$
(ii) $p_{0j}^h = p_{j0}^h = \delta_{hj} \qquad (0 \le h, j \le d)$
(iii) $k_h p_{ij}^h = k_j p_{i'h}^j \qquad (0 \le h, i, j \le d)$
(iv) $p_{i'j'}^{h'} = p_{ij}^h \qquad (0 \le h, i, j \le d)$
(v) $0' = 0$.

For the rest of this section, let $C = \langle X_0, \dots, X_d \rangle$ denote a character algebra. We now describe the algebraic structure of C. The scalar

$$N = \sum_{i=0}^{d} k_i \tag{7}$$

will play a role in our description; we observe by Definition 2.1(4) that $N \in \mathbb{R}^{>0}$. We refer to N as the *size* of C. By [9, Proposition 5.4, p. 92] there exists a basis $E_0, \ldots, E_d$ of C such that

$$E_i E_j = \delta_{ij} E_i \qquad (0 \le i, j \le d), \tag{8}$$

$$X_0 = \sum_{i=0}^{d} E_i, \tag{9}$$

and

$$E_0 = N^{-1} \sum_{i=0}^{d} X_i. \tag{10}$$

The basis $E_0, \ldots, E_d$ is unique up to a permutation of $E_1, \ldots, E_d$. We refer to the elements $E_0, \ldots, E_d$ as the *primitive idempotents* of C.

Since $X_0, \ldots, X_d$ and $E_0, \ldots, E_d$ are bases of C, there exist complex scalars $p_i(j)$ $(0 \le i, j \le d)$ such that

$$X_i = \sum_{j=0}^{d} p_i(j) E_j \qquad (0 \le i \le d) \tag{11}$$

and complex scalars $q_i(j)$ $(0 \le i, j \le d)$ such that

$$E_i = N^{-1} \sum_{j=0}^{d} q_i(j) X_j \qquad (0 \le i \le d). \tag{12}$$

Combining (11) and (8), we see that

$$X_i E_j = p_i(j) E_j \qquad (0 \le i, j \le d). \tag{13}$$

In view of (13), we refer to $p_i(j)$ as the ijth *eigenvalue* of C (with respect to the given ordering $E_0, \ldots, E_d$). We refer to $q_i(j)$ as the ijth *dual eigenvalue* of C (with respect to the given ordering $E_0, \ldots, E_d$).

Next we recall several identities involving the eigenvalues and dual eigenvalues of C. Combining (11) and (12) we find that

$$\sum_{j=0}^{d} p_i(j) q_j(h) = \delta_{ih} N \qquad (0 \le h, i \le d) \tag{14}$$

and

$$\sum_{j=0}^{d} q_i(j) p_j(h) = \delta_{ih} N \qquad (0 \le h, i \le d). \tag{15}$$

Setting $i = 0$ in (11) and comparing the result with (9) we see that

$$p_0(i) = 1 \qquad (0 \le i \le d). \tag{16}$$

Similarly, using (12) and (10) we find

$$q_0(i) = 1 \qquad (0 \leq i \leq d). \tag{17}$$

Applying π_0 to (10) and using (7) we find that $\pi_0(E_0) = 1$. Now setting $j = 0$ in (13) and applying π_0 to the result gives

$$p_i(0) = k_i \qquad (0 \leq i \leq d). \tag{18}$$

Motivated by (18), we write

$$m_i = q_i(0) \qquad (0 \leq i \leq d). \tag{19}$$

By [9, Theorem 5.5, p. 94] we have

$$p_i(j) m_j = \overline{q_j(i)} k_i \qquad (0 \leq i, j \leq d) \tag{20}$$

and

$$m_i \in \mathbb{R}^{>0} \qquad (0 \leq i \leq d). \tag{21}$$

We now recall several identities involving the complex conjugates of the eigenvalues and dual eigenvalues of C. By [9, p. 90] we have

$$p_{i'}(j) = \overline{p_i(j)} \qquad (0 \leq i, j \leq d); \tag{22}$$

combining this with (14) gives

$$q_i(j') = \overline{q_i(j)} \qquad (0 \leq i, j \leq d). \tag{23}$$

There exists a permutation $i \mapsto \hat{i}$ of $0, \dots, d$ such that $\tau(E_i) = E_{\hat{i}}$ for $0 \leq i \leq d$; it is routine to verify that this permutation is an involution and satisfies $\hat{0} = 0$. Using (11), (12), (22), and (23), one can also show that

$$p_i(\hat{j}) = \overline{p_i(j)} \qquad (0 \leq i, j \leq d) \tag{24}$$

and

$$q_{\hat{i}}(j) = \overline{q_i(j)} \qquad (0 \leq i, j \leq d). \tag{25}$$

We conclude our discussion of C by recalling how the structure constants of C are determined by the eigenvalues and dual eigenvalues of C. Namely, by [9, p. 96] we have

$$p_{ij}^h = N^{-1} \sum_{r=0}^{d} p_i(r) p_j(r) q_r(h) \qquad (0 \leq h, i, j \leq d). \tag{26}$$

Using (26) in combination with (15), one can also show that

$$p_i(r) p_j(r) = \sum_{h=0}^{d} p_{ij}^h p_h(r) \qquad (0 \leq i, j, r \leq d). \tag{27}$$

3. DUAL CHARACTER ALGEBRAS

In this section we describe the notion of duality for character algebras. The interested reader will find another account of these ideas in [9]. We begin by recalling the matrix of eigenvalues of a character algebra.

DEFINITION 3.1. Suppose $C = \langle X_0, \dots, X_d \rangle$ is a character algebra and fix an ordering $E_0, \dots, E_d$ of the primitive idempotents of C. Let P denote the matrix in $M_{d+1}(\mathbb{C})$ which satisfies

$$P_{ij} = p_j(i) \qquad (0 \le i, j \le d). \tag{28}$$

We refer to P as the *matrix of eigenvalues* of C associated with the ordering $E_0, \dots, E_d$.

We are now ready to define duality.

DEFINITION 3.2. Suppose $C = \langle X_0, \dots, X_d \rangle$ and $C^* = \langle X_0^*, \dots, X_d^* \rangle$ are character algebras. Fix an ordering $E_0, \dots, E_d$ of the primitive idempotents of C and let P denote the associated matrix of eigenvalues. Fix an ordering $E_0^*, \dots, E_d^*$ of the primitive idempotents of C^* and let P^* denote the associated matrix of eigenvalues. We say C and C^* are *dual* (with respect to the given orderings of their primitive idempotents) whenever

$$PP^* \in \operatorname{Span}\{I\}. \tag{29}$$

In this case the size N of C and the size N^* of C^* coincide and $PP^* = NI$.

We present two examples of duality.

EXAMPLE 3.3. Let M denote a Bose Mesner algebra and define $A_0, \dots, A_d$ and $E_0, \dots, E_d$ as in the Introduction. Let M^* denote the associated dual Bose Mesner algebra and define $A_0^*, \dots, A_d^*$ and $E_0^*, \dots, E_d^*$ as in the Introduction. By Examples 2.2 and 2.3, $M = \langle A_0, \dots, A_d \rangle$ and $M^* = \langle A_0^*, \dots, A_d^* \rangle$ are character algebras. Moreover, $E_0, \dots, E_d$ is an ordering of the primitive idempotents of M and $E_0^*, \dots, E_d^*$ is an ordering of the primitive idempotents of M^*. Recall from the Introduction that there exists an isomorphism $\rho\colon M \to M^*$ of vector spaces such that $E_i^* = \rho(A_i)$ and $A_i^* = n^{-1}\rho(E_i)$ for $0 \le i \le d$, where n is the order of M. It follows that the transition matrix from $A_0^*, \dots, A_d^*$ to $E_0^*, \dots, E_d^*$ is n^{-1} times the transition matrix from $E_0, \dots, E_d$ to $A_0, \dots, A_d$. Therefore, M and M^* are dual with respect to the orderings $E_0, \dots, E_d$ and $E_0^*, \dots, E_d^*$.

EXAMPLE 3.4 [13, pp. 26–28]. Suppose G is a finite group, let C denote the corresponding character algebra of Example 2.4, and let C^* denote the corresponding character algebra of Example 2.5. There is a natural one to one correspondence between the irreducible characters of G and the primitive idempotents of C; this correspondence induces an ordering of the primitive idempotents of C with respect to which C and C^* are dual in the sense of Definition 3.2. For this ordering, P is essentially the character table of G. More specifically, let $C_0, C_1, \dots, C_d$ denote the conjugacy classes of G as in Example 2.4 and let $\chi_0, \chi_1, \dots, \chi_d$ denote the irreducible complex characters of G as in Example 2.5. Then for $0 \leq i, j \leq d$ we have $P_{ij} = |C_j| d_i^{-1} \chi_i(g_j)$, where $d_i = \deg \chi_i$ and $g_j \in C_j$.

The following notational convention will be useful when we deal with dual character algebras.

Note 3.5. Suppose $C = \langle X_0, \dots, X_d \rangle$ and $C^* = \langle X_0^*, \dots, X_d^* \rangle$ are character algebras which are dual with respect to the orderings $E_0, \dots, E_d$ and $E_0^*, \dots, E_d^*$ of their primitive idempotents. If we write f to denote a particular object associated with C and $E_0, \dots, E_d$ then we write f^* to denote the corresponding object associated with C^* and $E_0^*, \dots, E_d^*$.

The next two propositions describe certain relationships between two character algebras which are dual. We omit their proofs, which are routine.

PROPOSITION 3.6. *Suppose $C = \langle X_0, \dots, X_d \rangle$ and $C^* = \langle X_0^*, \dots, X_d^* \rangle$ are character algebras which are dual with respect to the orderings $E_0, \dots, E_d$ and $E_0^*, \dots, E_d^*$ of their primitive idempotents. With reference to Note* 3.5,

(i) $p_i(j) = q_i^*(j) \qquad (0 \leq i, j \leq d)$

(ii) $q_i(j) = p_i^*(j) \qquad (0 \leq i, j \leq d)$

(iii) $k_i = m_i^* \qquad (0 \leq i \leq d)$

(iv) $m_i = k_i^* \qquad (0 \leq i \leq d)$.

PROPOSITION 3.7. *Suppose $C = \langle X_0, \dots, X_d \rangle$ and $C^* = \langle X_0^*, \dots, X_d^* \rangle$ are character algebras which are dual with respect to the orderings $E_0, \dots, E_d$ and $E_0^*, \dots, E_d^*$ of their primitive idempotents. Then*

(i) *the map $i \mapsto i'$ associated with C is the same as the map $i \mapsto \hat{i}$ associated with C^* and $E_0^*, \dots, E_d^*$;*

(ii) *the map $i \mapsto \hat{i}$ associated with C and $E_0, \dots, E_d$ is the same as the map $i \mapsto i'$ associated with C^*.*

The following convention describes the notation we will use for the maps given in Proposition 3.7.

Note 3.8. Suppose $C = \langle X_0, \dots, X_d \rangle$ and $C^* = \langle X_0^*, \dots, X_d^* \rangle$ are character algebras which are dual with respect to the orderings $E_0, \dots, E_d$ and $E_0^*, \dots, E_d^*$ of their primitive idempotents. For $0 \leq i \leq d$, we write i'

(resp. $\hat{i}$) to denote the image of i under the map discussed in Proposition 3.7(i) (resp. 3.7(ii)).

4. THE ALGEBRA $\mathscr{T}$

In this section we introduce the Terwilliger algebra $\mathscr{T}$ associated with a pair of dual character algebras.

DEFINITION 4.1. Suppose $C = \langle X_0, \dots, X_d \rangle$ and $C^* = \langle X_0^*, \dots, X_d^* \rangle$ are character algebras which are dual with respect to the orderings $E_0, \dots, E_d$ and $E_0^*, \dots, E_d^*$ of their primitive idempotents. With reference to Note 3.5, let $\mathscr{T}$ denote the associative $\mathbb{C}$-algebra with 1 which is generated by the symbols $x_0, \dots, x_d, x_0^*, \dots, x_d^*$ subject to the relations

(T1) $x_0 = x_0^*$

(T2) $x_i x_j = \sum_{h=0}^d p_{ij}^h x_h \qquad (0 \le i, j \le d)$

(T2*) $x_i^* x_j^* = \sum_{h=0}^d p_{ij}^{h*} x_h^* \qquad (0 \le i, j \le d)$

(T3) $e_h^* x_i e_j^* = 0 \qquad \text{if } p_{ij}^h = 0 \qquad (0 \le h, i, j \le d)$

(T3*) $e_h x_i^* e_j = 0 \qquad \text{if } p_{ij}^{h*} = 0 \qquad (0 \le h, i, j \le d)$.

The p_{ij}^h are the structure constants of C, as in Definition 2.1, and the p_{ij}^{h*} are the structure constants of C^*. The e_i and e_i^* are defined by

$$e_i = N^{-1} \sum_{j=0}^{d} q_i(j) x_j \qquad (0 \le i \le d) \tag{30}$$

and

$$e_i^* = N^{-1} \sum_{j=0}^{d} q_i^*(j) x_j^* \qquad (0 \le i \le d). \tag{31}$$

Here N is as in (7), the $q_i(j)$ are the dual eigenvalues of C as in (12), and the $q_i^*(j)$ are the dual eigenvalues of C^*.

We remark that $\mathscr{T}$ is invariant under a reversal of the roles of C and C^*; we will often take advantage of this fact in what follows.

5. TWO SUBALGEBRAS OF $\mathscr{T}$

In this section we consider the subspaces $\mathrm{Span}\{x_0, \dots, x_d\}$ and $\mathrm{Span}\{x_0^*, \dots, x_d^*\}$ of $\mathscr{T}$; these turn out to be subalgebras of $\mathscr{T}$ which are isomorphic to C and C^*, respectively. We begin with an observation regarding x_0 and x_0^*.

PROPOSITION 5.1. *With reference to Definition* 4.1, *we have* $x_0 = x_0^* = 1$.

Proof. By (T1) we have $x_0 = x_0^*$; write e to denote this element of $\mathcal{T}$. To show that $e = 1$, observe by (T2) and Proposition 2.6(ii) that $ex_i = x_i e = x_i$ for $0 \le i \le d$. Similarly, $ex_i^* = x_i^* e = x_i^*$ for $0 \le i \le d$. Therefore $e = 1$, since $x_0, \ldots, x_d, x_0^*, \ldots, x_d^*$ generate $\mathcal{T}$. ∎

We now turn our attention to a certain map from C to $\mathcal{T}$.

PROPOSITION 5.2. *With reference to Definition* 4.1, *let* ϕ: $C \to \mathcal{T}$ *denote the linear map which satisfies* $\phi(X_i) = x_i$ *for* $0 \le i \le d$. *Then*

(i) ϕ *is a* $\mathbb{C}$-*algebra homomorphism*;

(ii) $\phi(E_i) = e_i \qquad (0 \le i \le d)$.

We write $\mathcal{C}$ *to denote the image of* ϕ.

Proof. (i) By Proposition 5.1 we have $\phi(X_0) = 1$. To see that $\phi(XY) = \phi(X)\phi(Y)$ for all $X, Y \in C$, compare (5) and (T2) and recall that $X_0, \ldots, X_d$ is a basis for C.

(ii) Apply ϕ to (12) and compare the result with (30). ∎

After developing some theory concerning $\mathcal{T}$, we will show that ϕ is injective. For now, however, we content ourselves with using ϕ to obtain certain relations in $\mathcal{T}$.

PROPOSITION 5.3. *With reference to Definition* 4.1,

(i) $1 = \sum_{i=0}^{d} e_i$

(ii) $e_i e_j = \delta_{ij} e_i \qquad (0 \le i, j \le d)$

(iii) $x_i = \sum_{j=0}^{d} p_i(j) e_j \qquad (0 \le i \le d)$.

Proof. (i) Apply ϕ to (9) and use Propositions 5.1 and 5.2(ii).

(ii) Apply ϕ to (8) and use Proposition 5.2.

(iii) Apply ϕ to (11) and use Proposition 5.2(ii). ∎

Next we focus our attention on $\mathcal{C}$.

PROPOSITION 5.4. *With reference to Definition* 4.1,

(i) $\mathcal{C} = \mathrm{Span}\{x_0, \ldots, x_d\}$,

(ii) $\mathcal{C} = \mathrm{Span}\{e_0, \ldots, e_d\}$,

(iii) $\mathcal{C}$ *is a commutative subalgebra of* $\mathcal{T}$.

Proof. (i) This is immediate from the definition of ϕ.

(ii) This is immediate from Proposition 5.2(ii), since $E_0, \ldots, E_d$ span C.

(iii) This is immediate from Proposition 5.2(i) and the fact that C is commutative. ∎

Reversing the roles of C and C^* in the last three propositions, we obtain the following three propositions.

PROPOSITION 5.5. *With reference to Definition* 4.1, *let* $\phi^*\colon C^* \to \mathscr{T}$ *denote the linear map which satisfies* $\phi^*(X_i^*) = x_i^*$ *for* $0 \le i \le d$. *Then*

(i) ϕ^* *is a* $\mathbb{C}$-*algebra homomorphism*;

(ii) $\phi^*(E_i^*) = e_i^* \qquad (0 \le i \le d)$.

We write $\mathscr{C}^*$ *to denote the image of* ϕ^*.

PROPOSITION 5.6. *With reference to Definition* 4.1,

(i) $1 = \sum_{i=0}^{d} e_i^*$

(ii) $e_i^* e_j^* = \delta_{ij} e_i^* \qquad (0 \le i, j \le d)$

(iii) $x_i^* = \sum_{j=0}^{d} p_i^*(j) e_j^* \qquad (0 \le i \le d)$.

PROPOSITION 5.7. *With reference to Definition* 4.1,

(i) $\mathscr{C}^* = \mathrm{Span}\{x_0^*, \dots, x_d^*\}$,

(ii) $\mathscr{C}^* = \mathrm{Span}\{e_0^*, \dots, e_d^*\}$,

(iii) $\mathscr{C}^*$ *is a commutative subalgebra of* $\mathscr{T}$.

PROPOSITION 5.8. *With reference to Definition* 4.1, *the algebra* $\mathscr{T}$ *is generated by* $\mathscr{C} \cup \mathscr{C}^*$.

Proof. The elements $x_0, \dots, x_d, x_0^*, \dots, x_d^*$ are contained in $\mathscr{C} \cup \mathscr{C}^*$ and generate $\mathscr{T}$. ∎

6. AN INVOLUTION OF $\mathscr{T}$

In this section we describe an involution of $\mathscr{T}$ which will be useful in some of our later calculations.

PROPOSITION 6.1. *With reference to Definition* 4.1 *and Note* 3.8, *there exists a unique map* $\dagger\colon \mathscr{T} \to \mathscr{T}$ *such that*

(i) $\dagger(x + y) = \dagger(x) + \dagger(y) \qquad (x, y \in \mathscr{T})$

(ii) $\dagger(\alpha x) = \overline{\alpha}\dagger(x) \qquad (x \in \mathscr{T}, \alpha \in \mathbb{C})$

(iii) $\dagger(xy) = \dagger(y)\dagger(x) \qquad (x, y \in \mathscr{T})$

(iv) $\dagger(x_i) = x_{i'} \qquad (0 \le i \le d)$

(v) $\dagger(x_i^*) = x_{\hat{i}}^* \qquad (0 \le i \le d)$.

Moreover,

(vi) $\dagger(e_i) = e_i \qquad (0 \le i \le d)$

(vii) $\dagger(e_i^*) = e_i^* \qquad (0 \le i \le d)$

(viii) $\dagger$ *is an involution.*

Proof. By definition $\mathscr{T}$ is the quotient T/J, where T is the free associative $\mathbb{C}$-algebra with 1 generated by $x_0, \dots, x_d, x_0^*, \dots, x_d^*$ and J is the two sided ideal in T generated by the union of the sets

$$T_1 = \{x_0 - x_0^*\}$$

$$T_2 = \left\{ x_i x_j - \sum_{h=0}^{d} p_{ij}^h x_h \,|\, 0 \le i, j \le d \right\},$$

$$T_2^* = \left\{ x_i^* x_j^* - \sum_{h=0}^{d} p_{ij}^{h*} x_h^* \,|\, 0 \le i, j \le d \right\},$$

$$T_3 = \left\{ e_h^* x_i e_j^* \,|\, 0 \le h, i, j \le d, p_{ij}^h = 0 \right\},$$

and

$$T_3^* = \left\{ e_h x_i^* e_j \,|\, 0 \le h, i, j \le d, p_{ij}^{h*} = 0 \right\}.$$

Let $\ddagger$: $T \to T$ denote the map which satisfies (i)–(v). To show that $\ddagger$ induces a map $\dagger$: $\mathscr{T} \to \mathscr{T}$, we show that J is invariant under $\ddagger$. To do this, we show that $\ddagger$ permutes the elements of each of the sets T_1–T_3^*.

To show that

$$\ddagger(x_0 - x_0^*) = x_0 - x_0^*, \tag{32}$$

observe that $\ddagger(x_0) = x_{0'} = x_0$ in view of Proposition 2.6(v). Similarly we obtain $\ddagger(x_0^*) = x_0^*$; now (32) follows in view of (i) and (ii).

To show that $\ddagger$ permutes the elements of T_2, we show that

$$\ddagger\left(x_i x_j - \sum_{h=0}^{d} p_{ij}^h x_h \right) = x_{j'} x_{i'} - \sum_{h=0}^{d} p_{j'i'}^h x_h \qquad (0 \le i, j \le d). \tag{33}$$

To do this, let i, j be given and observe by (i) and (ii) that

$$\ddagger\left(x_i x_j - \sum_{h=0}^{d} p_{ij}^h x_h \right) = \ddagger(x_i x_j) - \ddagger\left(\sum_{h=0}^{d} p_{ij}^h x_h \right). \tag{34}$$

By (iii) and (iv) we have

$$\ddagger(x_i x_j) = x_{j'} x_{i'}. \tag{35}$$

On the other hand, by (i), (ii), (iv), and by Definition 2.1(3) we have

$$\ddagger\left(\sum_{h=0}^{d} p_{ij}^{h} x_h \right) = \sum_{h=0}^{d} p_{ij}^{h} x_{h'}. \tag{36}$$

Using Proposition 2.6(i), (iv) and the fact that $'$ is an involution we obtain

$$\sum_{h=0}^{d} p_{ij}^{h} x_{h'} = \sum_{h=0}^{d} p_{j'i'}^{h} x_h. \tag{37}$$

Now (33) follows by combining (34), (35), (36), and (37).

We have now shown that $\ddagger$ permutes the elements of T_2. Reversing the roles of C and C^* in the above argument, we see that $\ddagger$ also permutes the elements of T_2^*.

To show that $\ddagger$ permutes the elements of T_3, we show that for $0 \leq h, i, j \leq d$ we have

$$\ddagger(e_h^* x_i e_j^*) = e_j^* x_{i'} e_h^* \tag{38}$$

and

$$p_{ij}^{h} = 0 \qquad \text{if and only if } p_{i'h}^{j} = 0. \tag{39}$$

To show (38), we first show that

$$\ddagger(e_i) = e_i \qquad (0 \leq i \leq d). \tag{40}$$

To do this, apply $\ddagger$ to (30) and use (i), (ii), and (iv) to obtain

$$\ddagger(e_i) = N^{-1} \sum_{j=0}^{d} \overline{q_i(j)} x_{j'} \qquad (0 \leq i \leq d). \tag{41}$$

Now use (23) to eliminate $\overline{q_i(j)}$ on the right side of (41) and replace j' with j in the result, obtaining

$$\ddagger(e_i) = N^{-1} \sum_{j=0}^{d} q_i(j) x_j \qquad (0 \leq i \leq d). \tag{42}$$

Compare (42) and (30) to obtain (40). To show (38), combine (40) with (iii) and (iv).

To show (39), use Proposition 2.6(iii) and the fact that $k_i \neq 0$ for $0 \leq i \leq d$.

We have now shown (38) and (39), and it follows that $\ddagger$ permutes the elements of T_3. Reversing the roles of C and C^* in the above argument, we see that $\ddagger$ also permutes the elements of T_3^*.

We have now shown that ‡ permutes the elements of the sets T_1–T_3^*, and it follows that J is invariant under ‡. Therefore, ‡ induces a map †: $\mathscr{T} \to \mathscr{T}$ which satisfies (i)–(v). We saw in the above argument that ‡ satisfies (vi); it follows that the induced map † also satisfies (vi). Reversing the roles of C and C^* in (vi), we see that † satisfies (vii) as well. The fact that † is unique follows from (i)–(v) and the fact that $x_0, \dots, x_d, x_0^*, \dots, x_d^*$ generate $\mathscr{T}$.

To see that † is an involution, observe by (i)–(iii) that the map $\dagger^2$ is a $\mathbb{C}$-algebra homomorphism. Now use (iv) and the fact that $'$ is an involution to find that $\dagger^2(x_i) = x_i$ for $0 \le i \le d$. Similarly, we have $\dagger^2(x_i^*) = x_i^*$ for $0 \le i \le d$. We conclude that $\dagger^2$ is the identity map, since $x_0, \dots, x_d, x_0^*, \dots, x_d^*$ generate $\mathscr{T}$. ∎

7. SOME REDUCTION RULES

In this section we find several equations which allow one to write long products involving e_0 or e_0^* as linear combinations of shorter products; we call these equations reduction rules. We begin with four such rules.

PROPOSITION 7.1. *With reference to Definition* 4.1, *for* $0 \le i, j \le d$ *we have*

(i) $e_j^* x_i e_0^* = \delta_{ij} x_i e_0^*$,
(ii) $e_j^* e_i e_0^* = N^{-1} q_i(j) x_j e_0^*$,
(iii) $x_j^* x_i e_0^* = q_j(i) x_i e_0^*$,
(iv) $x_j^* e_i e_0^* = \sum_{h=0}^{d} p_{ij}^{h*} e_h e_0^*$.

Proof. Let i, j be given.

(i) If $i \ne j$, then $p_{i0}^{j} = 0$ by Proposition 2.6(ii) so $e_j^* x_i e_0^* = 0$ by (T3), as desired.

If $i = j$, then use Proposition 5.6(i) and the result in the previous case to obtain

$$\begin{aligned} x_i e_0^* &= \sum_{r=0}^{d} e_r^* x_i e_0^* \\ &= e_i^* x_i e_0^*, \end{aligned}$$

as desired.

(ii) By (30),

$$e_j^* e_i e_0^* = e_j^* \left(N^{-1} \sum_{r=0}^{d} q_i(r) x_r \right) e_0^*.$$

Apply (i) to each term of the sum to obtain the desired result.

(iii) By Proposition 5.6(iii),

$$x_j^* x_i e_0^* = \left(\sum_{r=0}^{d} p_j^*(r) e_r^* \right) x_i e_0^* .$$

Now apply (i) to each term of the sum and use Proposition 3.6(ii) to obtain the desired result.

(iv) By (30),

$$x_j^* e_i e_0^* = x_j^* \left(N^{-1} \sum_{r=0}^{d} q_i(r) x_r \right) e_0^* .$$

Apply (iii) to each term of the sum and use Proposition 5.3(iii) to obtain

$$x_j^* e_i e_0^* = \sum_{h=0}^{d} \left(N^{-1} \sum_{r=0}^{d} q_i(r) q_j(r) p_r(h) \right) e_h e_0^* .$$

Now use Proposition 3.6(i), (ii), the fact that $N = N^*$, and (26) to find that the quantity in parentheses is p_{ij}^{h*}, as desired. ∎

We now take advantage of the antiautomorphism † of Proposition 6.1 and our ability to reverse the roles of C and C^* to find more reduction rules in $\mathscr{T}$.

PROPOSITION 7.2. *With reference to Definition* 4.1 *and Note* 3.8, *for* $0 \le i, j \le d$ *we have*

(i) $e_0^* x_i e_j^* = \delta_{i'j} e_0^* x_i$,
(ii) $e_0^* e_i e_j^* = N^{-1} \overline{q_i(j)} e_0^* x_{j'}$,
(iii) $e_0^* x_i x_j^* = \overline{q_j(i)} e_0^* x_i$,
(iv) $e_0^* e_i x_j^* = \sum_{h=0}^{d} p_{i\hat{j}}^{h*} e_0^* e_h$.

Proof. (i) Apply † to Proposition 7.1(i) and replace i with i'.

(ii) Apply † to Proposition 7.1(ii).

(iii) Apply † to Proposition 7.1(iii), replace i with i', replace j with $\hat{j}$, and use (23) and (25).

(iv) Apply † to Proposition 7.1(iv), replace j with $\hat{j}$, and use the fact that $p_{ij}^{h*} \in \mathbb{R}$. ∎

PROPOSITION 7.3. *With reference to Definition* 4.1, *for* $0 \le i, j \le d$ *we have*

(i) $e_j x_i^* e_0 = \delta_{ij} x_i^* e_0$,
(ii) $e_j e_i^* e_0 = N^{-1} p_i(j) x_j^* e_0$,
(iii) $x_j x_i^* e_0 = p_j(i) x_i^* e_0$,
(iv) $x_j e_i^* e_0 = \sum_{h=0}^{d} p_{ij}^h e_h^* e_0$.

Proof. Reverse the roles of C and C^* in Proposition 7.1 and use Proposition 3.6(i). ∎

PROPOSITION 7.4. *With reference to Definition* 4.1 *and Note* 3.8, *for* $0 \le i, j \le d$ *we have*

(i) $e_0 x_i^* e_j = \delta_{\hat{i}j} e_0 x_i^*$,
(ii) $e_0 e_i^* e_j = N^{-1} \overline{p_i(j)} e_0 x_{\hat{j}}^*$,
(iii) $e_0 x_i^* x_j = \overline{p_j(i)} e_0 x_i^*$,
(iv) $e_0 e_i^* x_j = \sum_{h=0}^{d} p_{ij'}^h e_0 e_h^*$.

Proof. Reverse the roles of C and C^* in Proposition 7.2 and use Propositions 3.6(i) and 3.7. ∎

The next three propositions are useful special cases of the last four propositions.

PROPOSITION 7.5. *With reference to Definition* 4.1 *and Note* 3.8, *for* $0 \le j \le d$ *we have*

(i) $e_j^* e_0 e_0^* = N^{-1} x_j e_0^*$,
(ii) $e_0^* e_0 e_j^* = N^{-1} e_0^* x_{j'}$,
(iii) $e_j e_0^* e_0 = N^{-1} x_j^* e_0$,
(iv) $e_0 e_0^* e_j = N^{-1} e_0 x_{\hat{j}}^*$.

Proof. (i) Set $i = 0$ in Proposition 7.1(ii) and use (17).

(ii)–(iv) These are similar to the proof of (i). ∎

PROPOSITION 7.6. *With reference to Definition* 4.1 *and Note* 3.8, *for* $0 \le j \le d$ *we have*

(i) $x_j^* e_0 e_0^* = e_j e_0^*$,
(ii) $e_0^* e_0 x_j^* = e_0^* e_{\hat{j}}$,
(iii) $x_j e_0^* e_0 = e_j^* e_0$,
(iv) $e_0 e_0^* x_j = e_0 e_{j'}^*$.

Proof. (i) Set $i = 0$ in Proposition 7.1(iv) and use Proposition 2.6(ii).

(ii)–(iv) These are similar to the proof of (i). ∎

PROPOSITION 7.7. *With reference to Definition* 4.1,

(i) $e_0^* e_i e_0^* = N^{-1} m_i e_0^* \qquad (0 \leq i \leq d)$

(ii) $e_0 e_i^* e_0 = N^{-1} k_i e_0 \qquad (0 \leq i \leq d)$

(iii) $e_0^* e_0 e_0^* = N^{-1} e_0^*$

(iv) $e_0 e_0^* e_0 = N^{-1} e_0$.

Proof. (i) Set $j = 0$ in Proposition 7.1(ii), recall that $x_0 = 1$, and use (19).

(ii) Set $j = 0$ in Proposition 7.3(ii), recall that $x_0^* = 1$, and use (18).

(iii) Set $i = 0$ in (i) and use (17) and (19).

(iv) Set $i = 0$ in (ii) and use (16) and (18). ∎

So far in this section we have found reduction rules for all products of three symbols beginning or ending with either e_0 or e_0^*. We conclude this section by finding rules which relate three symbol products with e_0 in the center to three symbol products with e_0^* in the center.

PROPOSITION 7.8. *With reference to Definition* 4.1 *and Note* 3.8, *for* $0 \leq i, j \leq d$ *we have*

(i) $x_j e_0^* x_{i'} = N e_j^* e_0 e_i^*$,

(ii) $x_j e_0^* e_i = e_j^* e_0 x_{\hat{i}}^*$,

(iii) $e_j e_0^* x_{i'} = x_j^* e_0 e_i^*$,

(iv) $e_j e_0^* e_i = N^{-1} x_j^* e_0 x_{\hat{i}}^*$.

Proof. (i) Let i, j be given. By Proposition 7.7(iii) and the fact that $e_0^2 = e_0$, we have

$$x_j e_0^* x_{i'} = N x_j e_0^* e_0 e_0 e_0^* x_{i'}. \tag{43}$$

Now use Proposition 7.6(iii) to eliminate $x_j e_0^* e_0$ and Proposition 7.6(iv) to eliminate $e_0 e_0^* x_{i'}$ on the right side of (43), obtaining

$$x_j e_0^* x_{i'} = N e_j^* e_0 e_i^*,$$

as desired.

(ii) Let i, j be given. By Proposition 7.6(ii) and the fact that $\hat{}$ is an involution, we have

$$x_j e_0^* e_i = x_j e_0^* e_0 x_{\hat{i}}^*.$$

Now apply Proposition 7.6(iii) to $x_j e_0^* e_0$ to obtain

$$x_j e_0^* e_i = e_j^* e_0 x_{\hat{i}}^*,$$

as desired.

(iii) Reverse the roles of C and C^* in (ii) and use Proposition 3.7(i).

(iv) Reverse the roles of C and C^* in (i); now use Proposition 3.7(ii) and the fact that $N = N^*$. ∎

8. THE PRIMARY MODULE

With reference to Definition 4.1, let V denote a $\mathscr{T}$-module. By Proposition 5.3(i), (ii),

$$V = \sum_i e_i V \qquad (\text{direct sum}), \tag{44}$$

where the sum is over all i for which $e_i V \neq 0$. Similarly, by Proposition 5.6(i), (ii),

$$V = \sum_i e_i^* V \qquad (\text{direct sum}), \tag{45}$$

where the sum is over all i for which $e_i^* V \neq 0$. In this section we describe a $\mathscr{T}$-module for which the decompositions given in (44) and (45) are especially nice; we call this module the primary module.

LEMMA 8.1. *With reference to Definition* 4.1, *suppose V is a $\mathscr{T}$-module.*

(i) *For all nonzero $v \in e_0 V$ we have*

$$e_0 e_i^* v = N^{-1} k_i v \qquad (0 \leq i \leq d). \tag{46}$$

(ii) *For all nonzero $v \in e_0^* V$ we have*

$$e_0^* e_i v = N^{-1} m_i v \qquad (0 \leq i \leq d). \tag{47}$$

Proof. (i) Apply Proposition 7.7(ii) to v and observe that $v = e_0 v$.

(ii) Reverse the roles of C and C^* in (i) and use Proposition 3.6(iv). ∎

LEMMA 8.2. *With reference to Definition* 4.1, *suppose V is a $\mathscr{T}$-module.*

(i) *For all nonzero $v \in e_0 V$ the vectors $e_0^* v, \dots, e_d^* v$ are linearly independent.*

(ii) *For all nonzero $v \in e_0^* V$ the vectors $e_0 v, \dots, e_d v$ are linearly independent.*

Proof. (i) Use Lemma 8.1(i) and the fact that $N^{-1} k_i \neq 0$ to conclude that $e_i^* v \neq 0$ for $0 \leq i \leq d$. Now (i) follows routinely from Proposition 5.6(ii).

(ii) Reverse the roles of C and C^* in (i). ∎

Recall that a $\mathscr{T}$-module V is said to be *irreducible* if it is nonzero and its only $\mathscr{T}$-submodules are 0 and V.

PROPOSITION 8.3. *With reference to Definition* 4.1, *suppose V is a $\mathscr{T}$-module*; *then the following are equivalent*.

(i) $\dim e_i V = 1 \qquad (0 \le i \le d)$,

(ii) $e_0 V \neq 0$ *and V is irreducible*,

(iii) $\dim e_i^* V = 1 \qquad (0 \le i \le d)$,

(iv) $e_0^* V \neq 0$ *and V is irreducible*.

Proof. (i) $\Rightarrow$ (ii) Clearly $e_0 V \neq 0$. To show that V is irreducible, suppose W is a nonzero $\mathscr{T}$-submodule of V; we show that $W = V$.

By (i) and (44), we see that $\dim V = d + 1$. Fix a nonzero $v \in e_0 V$ and observe by Lemma 8.2(i) that the $d+1$ vectors $e_0^* v, \ldots, e_d^* v$ form a basis for V. Now fix a nonzero $w \in W$ and let $\alpha_0, \ldots, \alpha_d$ denote complex numbers such that

$$w = \sum_{i=0}^{d} \alpha_i e_i^* v. \tag{48}$$

Since $w \neq 0$, there exists j such that $\alpha_j \neq 0$. Apply e_j^* to both sides of (48) and use Proposition 5.6(ii) to obtain $\alpha_j^{-1} e_j^* w = e_j^* v$; it follows that $e_j^* v \in W$. Now use Lemma 8.1(i) and the fact that $N^{-1} k_i \neq 0$ to conclude that $v \in W$. It follows that $\{e_i^* v \mid 0 \le i \le d\} \subseteq W$, and since $e_0^* v, \ldots, e_d^* v$ form a basis for V, we have $W = V$ as desired.

(ii) $\Rightarrow$ (iii) Fix a nonzero $v \in e_0 V$; we show that $e_0^* v, \ldots, e_d^* v$ form a basis for V. By Lemma 8.2(i), the vectors $e_0^* v, \ldots, e_d^* v$ are linearly independent. To show that they span V, we show that $W = \operatorname{Span}\{e_i^* v \mid 0 \le i \le d\}$ is a $\mathscr{T}$-submodule of V.

By Proposition 7.3(iv) and the fact that $v = e_0 v$, we have

$$x_i e_j^* v = \sum_{h=0}^{d} p_{ij}^h e_h^* v \qquad (0 \le i, j \le d).$$

It follows that W is closed under $x_0, \ldots, x_d$. Since W is clearly closed under $x_0^*, \ldots, x_d^*$ and since the elements $x_0, \ldots, x_d, x_0^*, \ldots, x_d^*$ generate $\mathscr{T}$, we see that W is a $\mathscr{T}$-submodule of V, as desired.

Since V is irreducible and $W \neq 0$, it follows that $V = W$. Therefore, $e_0^* v, \ldots, e_d^* v$ form a basis for V and (iii) is apparent.

(iii) $\Rightarrow$ (iv) Reverse the roles of C and C^* in the proof of (i) $\Rightarrow$ (ii).

(iv) $\Rightarrow$ (i) Reverse the roles of C and C^* in the proof of (ii) $\Rightarrow$ (iii).

∎

PROPOSITION 8.4. *With reference to Definition* 4.1, *there exists a* $\mathcal{T}$*-module* V *which satisfies Proposition* 8.3(i)–(iv). *Moreover*, V *is unique up to isomorphism of* $\mathcal{T}$*-modules*. *We refer to* V *as the* primary module *of* $\mathcal{T}$.

Proof. To show existence, we construct a $\mathcal{T}$-module which satisfies Proposition 8.3(i)–(iv). Toward this end, let V denote a $d+1$ dimensional vector space over $\mathbb{C}$ and let $v_0, \ldots, v_d$ denote any basis of V. Let T denote the free associative $\mathbb{C}$-algebra with 1 generated by $x_0, \ldots, x_d, x_0^*, \ldots, x_d^*$. Give V the structure of a T-module such that

$$x_i v_j = p_i(j) v_j \qquad (0 \le i, j \le d) \tag{49}$$

and

$$x_i^* v_j = \sum_{h=0}^{d} p_{ij}^{h*} v_h \qquad (0 \le i, j \le d). \tag{50}$$

To show that V is a $\mathcal{T}$-module, we show that each of the relations of Definition 4.1(T1)–(T3*) hold on V. To do this, it is helpful to consider a second basis of V. Set

$$v_i^* = \sum_{j=0}^{d} p_i(j) v_j \qquad (0 \le i \le d) \tag{51}$$

and observe by (15) that

$$v_i = N^{-1} \sum_{j=0}^{d} q_i(j) v_j^* \qquad (0 \le i \le d). \tag{52}$$

By (52), the vectors $v_0^*, \ldots, v_d^*$ form a basis for V. We claim that

$$x_i^* v_j^* = q_i(j) v_j^* \qquad (0 \le i, j \le d) \tag{53}$$

and

$$x_i v_j^* = \sum_{h=0}^{d} p_{ij}^{h} v_h^* \qquad (0 \le i, j \le d). \tag{54}$$

To obtain (53), evaluate the left side using (51), (50), (52), Proposition 3.6(ii), (27), and (14) in that order. To obtain (54), use (51), (49), (52), and (26).

Combining (30), (49), and (15) we find

$$e_i v_j = \delta_{ij} v_j \qquad (0 \le i, j \le d). \tag{55}$$

Reversing the roles of C and C^* in (55), we obtain

$$e_i^* v_j^* = \delta_{ij} v_j^* \qquad (0 \le i, j \le d). \tag{56}$$

We now consider (T1)–(T3*).

(T1) We show that

$$(x_0 - x_0^*) v = 0 \qquad (v \in V). \tag{57}$$

Toward this end, set $i = 0$ in (49) and use (16) to see that x_0 acts as the identity on V. Reversing the roles of C and C^*, we find that x_0^* also acts as the identity on V, and (57) follows.

(T2) Let i, j be given with $0 \le i, j \le d$; we show that

$$\left(x_i x_j - \sum_{h=0}^{d} p_{ij}^h x_h \right) v = 0 \qquad (v \in V). \tag{58}$$

To do this, observe that for $0 \le r \le d$ we have

$$\begin{aligned} x_i x_j v_r &= p_i(r) p_j(r) v_r && (\text{by } (49)) \\ &= \sum_{h=0}^{d} p_{ij}^h p_h(r) v_r && (\text{by } (27)) \\ &= \sum_{h=0}^{d} p_{ij}^h x_h v_r && (\text{by } (49)). \end{aligned}$$

Now (58) follows.

(T2*) Let i, j be given with $0 \le i, j \le d$; we show that

$$\left(x_i^* x_j^* - \sum_{h=0}^{d} p_{ij}^{h*} x_h^* \right) v = 0 \qquad (v \in V). \tag{59}$$

To do this, reverse the roles of C and C^* in the proof of (T2) and use Proposition 3.6(ii).

(T3) Given h, i, j such that $p_{ij}^h = 0$, we show that

$$e_h^* x_i e_j^* v = 0 \qquad (v \in V). \tag{60}$$

To do this, observe that for $0 \le r \le d$ we have

$$\begin{aligned} e_h^* x_i e_j^* v_r^* &= \delta_{jr} e_h^* x_i v_r^* && (\text{by } (56)) \\ &= \delta_{jr} e_h^* \sum_{s=0}^{d} p_{ir}^h v_s^* && (\text{by } (54)) \\ &= \delta_{jr} p_{ir}^h v_h^* && (\text{by } (56)) \\ &= 0. \end{aligned}$$

Now (60) follows.

(T3*) Given h, i, j such that $p_{ij}^{h*} = 0$, we show that

$$e_h x_i^* e_j v = 0 \qquad (v \in V). \tag{61}$$

To do this, reverse the roles of C and C^* in the proof of (T3).

We have now shown that (T1)–(T3*) hold on V, so V is $\mathscr{T}$-module. Next we show that V satisfies Proposition 8.3(i)–(iv). By (56) we see that for $0 \le i \le d$, the vector v_i^* is a basis for $e_i^* V$, so $\dim e_i^* V = 1$. In particular, V satisfies Proposition 8.3(iii), so V satisfies Proposition 8.3(i)–(iv), as desired.

Concerning the uniqueness of V, suppose W is a $\mathscr{T}$-module which satisfies Proposition 8.3(i)–(iv). We show that V and W are $\mathscr{T}$-module isomorphic. Toward this end, fix a nonzero $w \in e_0^* W$ and observe by Lemma 8.2(ii), (44), and Proposition 8.3(i) that $e_0 w, \dots, e_d w$ form a basis for W. Using Proposition 5.3(ii), (iii) we obtain

$$x_i e_j w = p_i(j) e_j w \qquad (0 \le i, j \le d). \tag{62}$$

Using Proposition 7.1(iv) and the fact that $w = e_0^* w$, we obtain

$$x_i^* e_j w = \sum_{h=0}^{d} p_{ij}^{h*} e_h w \qquad (0 \le i, j \le d). \tag{63}$$

Comparing (62) with (49) and (63) with (50) we see that the linear map φ: $W \to V$ which has $\varphi(e_i w) = v_i$ for $0 \le i \le d$ is a $\mathscr{T}$-module isomorphism, as desired. ∎

9. TWO BASES FOR THE PRIMARY MODULE

In this section we describe two bases for the primary module with respect to which the action of the generators x_i and x_i^* of $\mathscr{T}$ is especially nice.

PROPOSITION 9.1. *With reference to Definition* 4.1, *let V denote the primary module for $\mathscr{T}$. Then for $v_0, \dots, v_d$ in V, the following are equivalent.*

(i) *There exists a nonzero $v \in e_0^* V$ such that*

$$v_i = e_i v \qquad (0 \le i \le d). \tag{64}$$

(ii) *At least one of $v_0, \dots, v_d$ is nonzero, $v_i \in e_i V$ for $0 \le i \le d$, and $\sum_{i=0}^{d} v_i \in e_0^* V$.*

Moreover, suppose (i) *and* (ii) *hold. Then* $v_0, \dots, v_d$ *is a basis for* V *and*

$$v = \sum_{i=0}^{d} v_i. \tag{65}$$

Proof. (i) ⇒ (ii) The vectors $v_0, \dots, v_d$ are linearly independent by Lemma 8.2(ii) so they are nonzero. Observe that $v_i = e_i v \in e_i V$ for $0 \le i \le d$. To obtain the last assertion, it suffices to prove (65). To obtain (65), sum (64) over i and use Proposition 5.3(i).

(ii) ⇒ (i) Let v be as in (65) and observe that $v \in e_0^* V$ by hypothesis. To see that (64) holds, apply e_i to (65) and use Proposition 5.3(ii) and the fact that $v_i \in e_i V$ for $0 \le i \le d$. Finally, observe that $v \neq 0$; otherwise $v_0, \dots, v_d$ are all zero by (64), which contradicts our hypothesis.

Now suppose (i) and (ii) hold. We have observed that $v_0, \dots, v_d$ are linearly independent, and they span V since $\dim V = d + 1$. Therefore $v_0, \dots, v_d$ forms a basis for V, as desired. We saw in the proof of (i) ⇒ (ii) that (65) holds. ∎

Motivated by Proposition 9.1, we make the following definition.

DEFINITION 9.2. With reference to Definition 4.1, let V denote the primary module for $\mathcal{T}$. By a *standard basis* of V we mean a sequence $v_0, \dots, v_d$ of vectors in V which satisfies Proposition 9.1(i), (ii).

PROPOSITION 9.3. *With reference to Definition* 4.1, *let* V *denote the primary module for* $\mathcal{T}$ *and suppose* $v_0, \dots, v_d$ *is a standard basis of* V. *Then for* $w_0, \dots, w_d$ *in* V, *the following are equivalent.*

(i) *There exists a nonzero* $\alpha \in \mathbb{C}$ *such that* $v_i = \alpha w_i$ *for* $0 \le i \le d$.

(ii) $w_0, \dots, w_d$ *is a standard basis of* V.

Proof. (i) ⇒ (ii) This is clear from Proposition 9.1(ii).

(ii) ⇒ (i) By Proposition 9.1(i) there exists a nonzero $v \in e_0^* V$ such that $v_i = e_i v$ for $0 \le i \le d$. Similarly, there exists a nonzero $w \in e_0^* V$ such that $w_i = e_i w$ for $0 \le i \le d$. Recall that $\dim e_0^* V = 1$ by Proposition 8.3(iii), so there exists a nonzero $\alpha \in \mathbb{C}$ such that $v = \alpha w$. Combining the above information, we find that $v_i = \alpha w_i$ for $0 \le i \le d$, as desired. ∎

In view of Proposition 9.3, we sometimes abuse language by referring to "the" standard basis of V.

We now describe how the elements x_i, x_i^*, e_i, and e_i^* of $\mathcal{T}$ act on the primary module with respect to the standard basis.

PROPOSITION 9.4. *With reference to Definition* 4.1, *let* V *denote the primary module for* $\mathcal{T}$ *and let* $v_0, \ldots, v_d$ *denote a standard basis of* V. *Then for* $0 \le i, j \le d$ *we have*

(i) $e_i v_j = \delta_{ij} v_j$,

(ii) $x_i v_j = p_i(j) v_j$,

(iii) $e_i^* v_j = N^{-1} q_j(i) \sum_{r=0}^{d} p_i(r) v_r$,

(iv) $x_i^* v_j = \sum_{r=0}^{d} p_{ij}^{r*} v_r$.

Proof. (i) This is immediate from (64) and Proposition 5.3(ii).

(ii) Use Proposition 5.3(iii) to eliminate x_i on the left and use (i) to evaluate the result.

(iii) By Proposition 9.1(i) there exists a nonzero $v \in e_0^* V$ such that $v_i = e_i v$ for $0 \le i \le d$. By Proposition 5.6(ii) we have $v = e_0^* v$. Therefore,

$$
\begin{aligned}
e_i^* v_j &= e_i^* e_j v \\
&= e_i^* e_j e_0^* v \\
&= N^{-1} q_j(i) x_i v && \text{(by Proposition 7.1(ii))} \\
&= N^{-1} q_j(i) \sum_{r=0}^{d} p_i(r) e_r v && \text{(by Proposition 5.3(iii))} \\
&= N^{-1} q_j(i) \sum_{r=0}^{d} p_i(r) v_r
\end{aligned}
$$

as desired.

(iv) By Proposition 9.1(i) there exists a nonzero $v \in e_0^* V$ such that $v_i = e_i v$ for $0 \le i \le d$. By Proposition 5.6(ii) we have $v = e_0^* v$. Now apply both sides of Proposition 7.1(iv) to v and use the above information to evaluate the result. ∎

Reversing the roles of C and C^* in the above propositions, we obtain the following propositions.

PROPOSITION 9.5. *With reference to Definition* 4.1, *let* V *denote the primary module for* $\mathcal{T}$. *Then for* $v_0^*, \ldots, v_d^*$ *in* V, *the following are equivalent.*

(i) *There exists a nonzero* $v \in e_0 V$ *such that*

$$v_i^* = e_i^* v \qquad (0 \le i \le d). \tag{66}$$

(ii) *At least one of* $v_0^*, \ldots, v_d^*$ *is nonzero*, $v_i^* \in e_i^* V$ *for* $0 \le i \le d$, *and* $\sum_{i=0}^{d} v_i^* \in e_0 V$.

Moreover, suppose (i) *and* (ii) *hold. Then* $v_0^*, \ldots, v_d^*$ *is a basis for* V *and*

$$v = \sum_{i=0}^{d} v_i^*. \tag{67}$$

DEFINITION 9.6. With reference to Definition 4.1, let V denote the primary module for $\mathscr{T}$. By a *dual standard basis* of V we mean a sequence $v_0^*, \ldots, v_d^*$ of vectors in V which satisfies Proposition 9.5(i), (ii).

PROPOSITION 9.7. *With reference to Definition* 4.1, *let* V *denote the primary module for* $\mathscr{T}$ *and suppose* $v_0^*, \ldots, v_d^*$ *is a dual standard basis of* V. *Then for* $w_0^*, \ldots, w_d^*$ *in* V, *the following are equivalent.*

(i) *There exists a nonzero* $\alpha \in \mathbb{C}$ *such that* $v_i^* = \alpha w_i^*$ *for* $0 \le i \le d$.

(ii) $w_0^*, \ldots, w_d^*$ *is a dual standard basis of* V

PROPOSITION 9.8. *With reference to Definition* 4.1, *let* V *denote the primary module for* $\mathscr{T}$ *and let* $v_0^*, \ldots, v_d^*$ *denote a dual standard basis of* V. *Then for* $0 \le i, j \le d$ *we have*

(i) $e_i^* v_j^* = \delta_{ij} v_j^*$,

(ii) $x_i^* v_j^* = p_i^*(j) v_j^*$,

(iii) $e_i v_j^* = N^{-1} q_j^*(i) \sum_{r=0}^{d} p_i^*(r) v_r^*$,

(iv) $x_i v_j^* = \sum_{r=0}^{d} p_{ij}^r v_r^*$.

We conclude our discussion of the standard and dual standard bases by determining the transition matrices between them.

PROPOSITION 9.9. *With reference to Definition* 4.1, *let* V *denote the primary module for* $\mathscr{T}$ *and let* $v_0, \ldots, v_d$ *denote a standard basis of* V. *Then there exists a dual standard basis* $v_0^*, \ldots, v_d^*$ *of* V *such that*

$$v_i^* = \sum_{j=0}^{d} p_i(j) v_j \qquad (0 \le i \le d) \tag{68}$$

and

$$v_i = N^{-1} \sum_{j=0}^{d} q_i(j) v_j^* \qquad (0 \le i \le d). \tag{69}$$

Proof. Write $v_i^* = N e_i^* v_0$ for $0 \le i \le d$. Then Proposition 9.5(i) holds with $v = N v_0$ so $v_0^*, \ldots, v_d^*$ is a dual standard basis of V. To show (68), set $j = 0$ in Proposition 9.4(iii) and use (17). To show (69), use (68), and (15). ∎

10. TWO SUBALGEBRAS OF $\mathcal{T}$, REVISITED

In this section we return our attention to the subalgebras $\mathcal{C}$ and $\mathcal{C}^*$ of $\mathcal{T}$ which are discussed in Section 5.

LEMMA 10.1. *With reference to Definition* 4.1, *the elements* $x_0, \ldots, x_d, x_1^*, \ldots, x_d^*$ *of* $\mathcal{T}$ *are linearly independent*. (*Recall from Definition* 4.1(T1) *that* $x_0 = x_0^*$.)

Proof. Suppose complex numbers $\alpha_0, \ldots, \alpha_d, \alpha_1^*, \ldots, \alpha_d^*$ are given such that

$$\alpha_0 x_0 + \sum_{i=1}^{d} \alpha_i x_i + \sum_{i=1}^{d} \alpha_i^* x_i^* = 0. \tag{70}$$

We show that $\alpha_0, \ldots, \alpha_d, \alpha_1^*, \ldots, \alpha_d^*$ are all zero. Toward this end, we first show that $\alpha_r = 0$ for $1 \leq r \leq d$. Let r be given and multiply each side of (70) on the left by e_r^* and on the right by e_0^*. Use Propositions 5.6(ii), 5.7(iii), and 7.1(i) to simplify the result, obtaining

$$\alpha_r x_r e_0^* = 0. \tag{71}$$

We claim that $x_r e_0^* \neq 0$. To see this, let V denote the primary module for $\mathcal{T}$ and let $v_0^*, \ldots, v_d^*$ denote a dual standard basis for V. Set $j = 0$ in Proposition 9.8(iv) and use Proposition 2.6(ii) to evaluate the result, obtaining $x_r v_0^* = v_r^*$. By Proposition 9.8(i) we have $e_0^* v_0^* = v_0^*$, and it follows that $x_r e_0^* v_0^* = v_r^*$. Since $v_r^* \neq 0$, we must have $x_r e_0^* \neq 0$, as claimed. Combining this with (71), we find that $\alpha_r = 0$.

Reversing the roles of C and C^* in the above argument, we find that $\alpha_1^*, \ldots, \alpha_d^*$ are zero; now (70) reduces to $\alpha_0 x_0 = 0$. Setting $i = j = 0$ in Proposition 9.8(iv) and using Proposition 2.6(ii) to simplify the result, we find that $x_0 v_0^* = v_0^*$. Therefore $x_0 \neq 0$ and it follows that $\alpha_0 = 0$, as desired. ∎

PROPOSITION 10.2. *With reference to Definition* 4.1,

(i) *the map* ϕ *of Proposition* 5.2 *is injective*,

(ii) $\dim \mathcal{C} = d + 1$,

(iii) $x_0, \ldots, x_d$ *is a basis of* $\mathcal{C}$,

(iv) $e_0, \ldots, e_d$ *is a basis of* $\mathcal{C}$.

Proof. (i) This is immediate from Lemma 10.1.

(ii) This is immediate from (i) and the fact that $\dim C = d + 1$.

(iii) This is immediate from (ii) and Proposition 5.4(i).

(iv) This is immediate from (ii) and Proposition 5.4(ii). ∎

PROPOSITION 10.3. *With reference to Definition* 4.1,

(i) *the map* ϕ^* *of Proposition* 5.5 *is injective*,

(ii) $\dim \mathscr{C}^* = d + 1$,

(iii) $x_0^*, \dots, x_d^*$ *is a basis of* $\mathscr{C}^*$,

(iv) $e_0^*, \dots, e_d^*$ *is a basis of* $\mathscr{C}^*$.

Proof. Reverse the roles of C and C^* in Proposition 10.2. ∎

PROPOSITION 10.4. *With reference to Definition* 4.1, *we have* $\mathscr{C} \cap \mathscr{C}^* =$ Span$\{1\}$.

Proof. Recall from Proposition 5.1 that $1 = x_0 \in \mathscr{C}$ and $1 = x_0^* \in \mathscr{C}^*$, so Span$\{1\} \subseteq \mathscr{C} \cap \mathscr{C}^*$. The fact that $\mathscr{C} \cap \mathscr{C}^* \subseteq$ Span$\{1\}$ is immediate from Propositions 10.2(iii), 10.3(iii), and Lemma 10.1. ∎

11. A CENTRAL IDEMPOTENT OF $\mathscr{T}$

We now turn our attention to the algebraic structure of $\mathscr{T}$. In this section we introduce an element u_0 of $\mathscr{T}$ and we use the reduction rules of Section 7 to show that u_0 is a central idempotent. We begin with another relation in $\mathscr{T}$.

PROPOSITION 11.1. *With reference to Definition* 4.1,

$$N \sum_{r=0}^{d} k_r^{-1} e_r^* e_0 e_r^* = N \sum_{j=0}^{d} k_j^{*-1} e_j e_0^* e_j. \tag{72}$$

We write u_0 *to denote this element of* $\mathscr{T}$.

Proof. We have

$$
\begin{aligned}
N \sum_{j=0}^{d} k_j^{*-1} e_j e_0^* e_j
&= \sum_{j=0}^{d} k_j^{*-1} x_j^* e_0 x_j^* && (\text{by Proposition 7.8(iv)}) \\
&= \sum_{j=0}^{d} \sum_{r=0}^{d} \sum_{s=0}^{d} k_j^{*-1} p_j^*(s) p_j^*(r) e_r^* e_0 e_s^* && (\text{by Proposition 5.6(iii)}) \\
&= \sum_{j=0}^{d} \sum_{r=0}^{d} \sum_{s=0}^{d} k_j^{*-1} \overline{p_j^*(s)} p_j^*(r) e_r^* e_0 e_s^* \\
& && (\text{by Proposition 3.7(i) and (22)}) \\
&= \sum_{r=0}^{d} \sum_{s=0}^{d} m_s^{*-1} \sum_{j=0}^{d} q_s^*(j) p_j^*(r) e_r^* e_0 e_s^* && (\text{by (20)}) \\
&= N \sum_{r=0}^{d} m_r^{*-1} e_r^* e_0 e_r^* && (\text{by (15)}) \\
&= N \sum_{r=0}^{d} k_r^{-1} e_r^* e_0 e_r^* && (\text{by Proposition 3.6(iii)}),
\end{aligned}
$$

as desired. ∎

Next we consider certain products involving u_0.

PROPOSITION 11.2. *With reference to Definition* 4.1, *for* $0 \le i \le d$ *we have*

(i) $e_i u_0 = N k_i^{*-1} e_i e_0^* e_i$,
(ii) $u_0 e_i = N k_i^{*-1} e_i e_0^* e_i$,
(iii) $e_i^* u_0 = N k_i^{-1} e_i^* e_0 e_i^*$
(iv) $u_0 e_i^* = N k_i^{-1} e_i^* e_0 e_i^*$.

Proof. (i), (ii) Use the right side of (72) to eliminate u_0 and use Proposition 5.3(ii) to simplify the result.

(iii), (iv) Use the left side of (72) to eliminate u_0 and use Proposition 5.6(ii) to simplify the result. ∎

The following corollary of Proposition 11.2 will be useful later on.

COROLLARY 1.13. *With reference to Definition* 4.1,

(i) $e_0 = e_0 u_0$,

(ii) $e_0^* = e_0^* u_0$,

(iii) $u_0 \neq 0$.

Proof. (i) Set $i = 0$ in Proposition 11.2(i) and use (16), (18), and Proposition 7.7(iv) to evaluate the result.

(ii) This is similar to the proof of (i).

(iii) This is immediate from (i) and Proposition 10.2(iv). ∎

Next we show that u_0 is a central idempotent of $\mathscr{T}$.

PROPOSITION 11.4. *With reference to Definition* 4.1,

(i) $u_0 t = t u_0 \qquad (t \in \mathscr{T})$,

(ii) $u_0^2 = u_0$.

In other words, u_0 *is a central idempotent of* $\mathscr{T}$.

Proof. (i) By Proposition 11.2, the element u_0 commutes with the elements $e_0, \ldots, e_d, e_0^*, \ldots, e_d^*$. These elements generate $\mathscr{T}$ by Propositions 5.4(ii), 5.7(ii), and 5.8, so (i) holds.

(ii) We have

$$\begin{aligned}
u_0^2 &= u_0 \left(N \sum_{i=0}^{d} k_i^{-1} e_i^* e_0 e_i^* \right) && (\text{by } (72)) \\
&= N^2 \sum_{i=0}^{d} k_i^{-2} e_i^* e_0 e_i^* e_0 e_i^* && (\text{by Proposition } 11.2(\text{iv})) \\
&= N \sum_{i=0}^{d} k_i^{-1} e_i^* e_0 e_i^* && (\text{by Proposition } 7.7(\text{ii})) \\
&= u_0,
\end{aligned}$$

as desired. ∎

We conclude this section by finding a decomposition of $\mathscr{T}$ as a direct sum of two sided ideals. We omit the proof, which is routine.

PROPOSITION 11.5. *With reference to Definition* 4.1,

(i) $\mathcal{T}u_0$ *is a nonzero two sided ideal of* $\mathcal{T}$,

(ii) $\mathcal{T}(1-u_0)$ *is a two sided ideal of* $\mathcal{T}$,

(iii) $\mathcal{T}=\mathcal{T}u_0+\mathcal{T}(1-u_0)$ *and the sum is direct*.

We remark that in the Introduction we wrote $\mathcal{T}_0$ and $\mathcal{T}_1$ to denote the ideals $\mathcal{T}u_0$ and $\mathcal{T}(1-u_0)$, respectively.

12. THE IDEAL $\mathcal{T}u_0$

In view of Proposition 11.5, we now restrict our attention to the ideal $\mathcal{T}u_0$. By Proposition 11.4(ii), the ideal $\mathcal{T}u_0$ is a $\mathbb{C}$-algebra with identity u_0; in this section we show that $\mathcal{T}u_0$ is $\mathbb{C}$-algebra isomorphic to the full matrix algebra $M_{d+1}(\mathbb{C})$. We begin by describing several ways to recognize $\mathcal{T}u_0$.

PROPOSITION 12.1. *With reference to Definition* 4.1, *and using the notation of Propositions* 5.2 *and* 5.5, *the vector spaces* $\mathcal{C}e_0^*\mathcal{C}$, $\mathcal{C}^*e_0\mathcal{C}^*$, $\mathcal{T}e_0\mathcal{T}$, *and* $\mathcal{T}e_0^*\mathcal{T}$ *are all equal to* $\mathcal{T}u_0$.

Proof. First observe that

$$\begin{aligned}
\mathcal{C}e_0^*\mathcal{C} &= \operatorname{Span}\{x_j e_0^* x_{i'} \mid 0 \le i, j \le d\} && \text{(by Proposition 5.4(i))}\\
&= \operatorname{Span}\{e_j^* e_0 e_i^* \mid 0 \le i, j \le d\} && \text{(by Proposition 7.8(i))}\\
&= \mathcal{C}^* e_0 \mathcal{C}^* && \text{(by Proposition 5.7(ii))}.
\end{aligned}$$

For the rest of the proof we write $T_0=\mathcal{C}e_0^*\mathcal{C}=\mathcal{C}^*e_0\mathcal{C}^*$.

By Proposition 5.8 the set $\mathcal{C}\cup\mathcal{C}^*$ generates $\mathcal{T}$, so we have $\mathcal{T}T_0\subseteq T_0$ and $T_0\mathcal{T}\subseteq T_0$; it follows that T_0 is a two sided ideal of $\mathcal{T}$. To see that $T_0=\mathcal{T}e_0\mathcal{T}$, first observe that by construction $T_0\subseteq\mathcal{T}e_0\mathcal{T}$. To obtain the reverse inclusion, observe that $e_0\in T_0$, so

$$\begin{aligned}
\mathcal{T}e_0\mathcal{T} &\subseteq \mathcal{T}T_0\mathcal{T}\\
&\subseteq T_0.
\end{aligned}$$

Therefore, $T_0=\mathcal{T}e_0\mathcal{T}$.

To see that $T_0=\mathcal{T}e_0^*\mathcal{T}$, reverse the roles of C and C^* in the above argument.

It remains to show that $T_0=\mathcal{T}u_0$. To do this, use Corollary 11.3(i) to find that $e_0\in\mathcal{T}u_0$, and conclude that $T_0=\mathcal{T}e_0\mathcal{T}\subseteq\mathcal{T}u_0$. From the form of u_0 in (72) we see that $u_0\in\mathcal{C}^*e_0\mathcal{C}^*=T_0$, so $\mathcal{T}u_0\subseteq T_0$. We now have $T_0=Tu_0$, as desired. ∎

In the next proposition we present a basis for $\mathscr{T}u_0$ and compute products of these basic elements.

PROPOSITION 12.2. *With reference to Definition* 4.1, *write*

$$\Upsilon_{ij} = N e_i e_0^* e_j \qquad (0 \leq i, j \leq d). \tag{73}$$

Then

(i) $\{\Upsilon_{ij} \mid 0 \leq i, j \leq d\}$ *is a basis for* $\mathscr{T}u_0$,

(ii) $\Upsilon_{ij}\Upsilon_{rs} = \delta_{jr} m_j \Upsilon_{is} \qquad (0 \leq i, j, r, s \leq d)$.

Proof. (i) We first show that $\Upsilon_{ij} \neq 0$ for $0 \leq i, j \leq d$. Toward this end, let i, j be given and use Proposition 7.7(i) and the fact that $e_0^{*2} = e_0^*$ to obtain

$$e_0^* \Upsilon_{ij} e_0^* = N^{-1} m_i m_j e_0^* \qquad (0 \leq i, j \leq d).$$

Since $N^{-1} m_i m_j \neq 0$ and $e_0^* \neq 0$, we conclude that $\Upsilon_{ij} \neq 0$. It is now routine using Proposition 5.3(ii) to show that $\{\Upsilon_{ij} \mid 0 \leq i, j \leq d\}$ is linearly independent. To see that these elements span $\mathscr{T}u_0$, observe that

$$\begin{aligned}
&\operatorname{Span}\{\Upsilon_{ij} \mid 0 \leq i, j \leq d\} \\
&\quad = \operatorname{Span}\{e_i e_0^* e_j \mid 0 \leq i, j \leq d\} && (\text{by } (73)) \\
&\quad = \mathscr{C} e_0^* \mathscr{C} && (\text{by Proposition } 5.4(\text{ii})) \\
&\quad = \mathscr{T}u_0 && (\text{by Proposition } 12.1),
\end{aligned}$$

as desired.

(ii) Use (73) and Proposition 5.3(ii) to obtain

$$\Upsilon_{ij}\Upsilon_{rs} = \delta_{jr} N^2 e_i e_0^* e_j e_0^* e_s \qquad (0 \leq i, j, r, s \leq d).$$

Now use Proposition 7.7(i) to eliminate $e_0^* e_j e_0^*$ and compare the result with the right side of (73). ∎

We are now ready to describe the $\mathbb{C}$-algebra structure of $\mathscr{T}u_0$.

THEOREM 12.3. *With reference to Definition* 4.1, *there exists a* $\mathbb{C}$-*algebra isomorphism* $\rho: \mathscr{T}u_0 \to M_{d+1}(\mathbb{C})$ *such that*

$$\rho(\Upsilon_{ij}) = m_j e_{ij} \qquad (0 \leq i, j \leq d). \tag{74}$$

Proof. By Proposition 12.2(i), there exists a unique isomorphism of vector spaces $\rho: \mathscr{T}u_0 \to M_{d+1}(\mathbb{C})$ which satisfies (74). By Proposition 12.2(ii), this map is an isomorphism of $\mathbb{C}$-algebras. ∎

Reversing the roles of C and C^* in the previous two results, we obtain the following.

PROPOSITION 12.4. *With reference to Definition* 4.1, *write*

$$\Upsilon_{ij}^* = N e_i^* e_0 e_j^* \qquad (0 \leq i, j \leq d). \tag{75}$$

Then

(i) $\{\Upsilon_{ij}^* \mid 0 \leq i, j \leq d\}$ *is a basis for* $\mathscr{T} u_0$,

(ii) $\Upsilon_{ij}^* \Upsilon_{rs}^* = \delta_{jr} m_j^* \Upsilon_{is}^* \qquad (0 \leq i, j, r, s \leq d)$.

THEOREM 12.5. *With reference to Definition* 4.1, *there exists a* $\mathbb{C}$*-algebra isomorphism* $\rho^*: \mathscr{T} u_0 \to M_{d+1}(\mathbb{C})$ *such that*

$$\rho^*\left(\Upsilon_{ij}^*\right) = m_j^* e_{ij} \qquad (0 \leq i, j \leq d). \tag{76}$$

13. THE IDEAL $\mathscr{T} u_0$ AND THE PRIMARY MODULE

In this section we consider the connection between the ideal $\mathscr{T} u_0$ and the primary module. We begin by describing the action of the basis elements Υ_{ij} of Proposition 12.2 on the primary module.

PROPOSITION 13.1. *With reference to Definition* 4.1, *let V denote the primary module for $\mathscr{T}$ and let $v_0, \ldots, v_d$ denote a standard basis of V. Then the elements Υ_{ij} of* (73) *satisfy*

$$\Upsilon_{ij} v_r = \delta_{jr} m_j v_i \qquad (0 \leq i, j, r \leq d). \tag{77}$$

Proof. We have

$$\begin{aligned}
\Upsilon_{ij} v_r &= N e_i e_0^* e_j v_r && (\text{by } (73)) \\
&= \delta_{jr} N e_i e_0^* v_j && (\text{by Proposition } 9.4(\text{i})) \\
&= \delta_{jr} m_j e_i \sum_{s=0}^{d} v_s && (\text{by Proposition } 9.4(\text{iii}), (16), \text{and } (19)) \\
&= \delta_{jr} m_j v_i && (\text{by Proposition } 9.4(\text{i})),
\end{aligned}$$

as desired. ∎

Reversing the roles of C and C^* in the previous proposition, we obtain the following.

PROPOSITION 13.2. *With reference to Definition* 4.1, *let V denote the primary module for $\mathscr{T}$ and let $v_0^*, \ldots, v_d^*$ denote a dual standard basis of V.*

Then the elements Υ_{ij}^* *of* (75) *satisfy*

$$\Upsilon_{ij}^* v_r^* = \delta_{jr} m_j^* v_i^* \qquad (0 \le i, j, r \le d). \tag{78}$$

Before stating the main theorem of this section, we recall two definitions.

DEFINITION 13.3. With reference to Definition 4.1, let V denote the primary module for $\mathcal{T}$. We write $\mathrm{End}(V)$ to denote the $\mathbb{C}$-algebra of all linear maps from V to V. For $0 \le i, j \le d$, we write f_{ij} to denote the element of $\mathrm{End}(V)$ which has

$$f_{ij}(v_r) = \delta_{jr} v_i \qquad (0 \le r \le d), \tag{79}$$

where $v_0, \dots, v_d$ is a standard basis of V. We observe that $\{f_{ij} \mid 0 \le i, j \le d\}$ is a basis for $\mathrm{End}(V)$. Similarly, for $0 \le i, j \le d$ we write f_{ij}^* to denote the element of $\mathrm{End}(V)$ which has

$$f_{ij}^*(v_r^*) = \delta_{jr} v_i^* \qquad (0 \le r \le d), \tag{80}$$

where $v_0^*, \dots, v_d^*$ is a dual standard basis of V. We observe that $\{f_{ij}^* \mid 0 \le i, j \le d\}$ is a basis for $\mathrm{End}(V)$.

DEFINITION 13.4. With reference to Definition 4.1, let V denote the primary module for $\mathcal{T}$. Since V is a $\mathcal{T}$-module, there exists a natural $\mathbb{C}$-algebra homomorphism from $\mathcal{T}$ to $\mathrm{End}(V)$. We write η to denote this map, and we observe that

$$\eta(t)(v) = tv \qquad (t \in \mathcal{T}, v \in V). \tag{81}$$

We conclude this section by describing the action of η on the ideal $\mathcal{T}u_0$ and $\mathcal{T}(1 - u_0)$.

THEOREM 13.5. *With reference to Definition* 4.1, *let V denote the primary module for $\mathcal{T}$ and let η: $\mathcal{T} \to \mathrm{End}(V)$ denote the map of Definition* 13.4. *Then the restriction of η to the ideal $\mathcal{T}u_0$ is an isomorphism of $\mathbb{C}$-algebras and* $\ker \eta = \mathcal{T}(1 - u_0)$. *Moreover,*

$$\eta(\Upsilon_{ij}) = m_j f_{ij} \qquad (0 \le i, j \le d) \tag{82}$$

and

$$\eta(\Upsilon_{ij}^*) = m_j^* f_{ij}^* \qquad (0 \le i, j \le d). \tag{83}$$

Proof. Line (82) is immediate from (77), (79), and (81); line (83) is immediate from (78), (80), and (81). Let η_0 denote the restriction of η to $\mathcal{T}u_0$ and observe that η_0 is a homomorphism of $\mathbb{C}$-algebras. Combining

(82), the fact that $\{\Upsilon_{ij} \mid 0 \le i, j \le d\}$ is a basis for $\mathscr{T}u_0$, and the fact that $\{f_{ij} \mid 0 \le i, j \le d\}$ is a basis for $\mathrm{End}(V)$, we see that η_0 is a bijection. It follows that η_0 is an isomorphism of $\mathbb{C}$-algebras. It is routine to show $\ker \eta = \mathscr{T}(1 - u_0)$. ∎

14. MODULES FOR $\mathscr{T}$

In this section we investigate arbitrary $\mathscr{T}$-modules. We begin by using the central idempotent u_0 to write an arbitrary $\mathscr{T}$-module as a direct sum of two submodules. We omit the proof, which is routine.

Proposition 14.1. *With reference to Definition* 4.1, *suppose W is a $\mathscr{T}$-module. Then*

(i) *u_0W is a $\mathscr{T}$-submodule of W,*
(ii) *$(1 - u_0)W$ is a $\mathscr{T}$-submodule of W,*
(iii) *$W = u_0W + (1 - u_0)W$ and the sum is direct.*

In the next proposition we give several ways of recognizing u_0W.

Proposition 14.2. *With reference to Definition* 4.1, *suppose W is a $\mathscr{T}$-module. Then each of $\mathscr{T}e_0W$, $\mathscr{T}e_0^*W$, $\mathscr{C}e_0^*W$, and $\mathscr{C}^*e_0W$ is equal to u_0W.*

Proof. Observe that $u_0 \in \mathscr{C}^*e_0\mathscr{C}^*$ by (72), and $\mathscr{C}^*W \subseteq W$, so $u_0W \subseteq \mathscr{C}^*e_0W$. Clearly $\mathscr{C}^*e_0W \subseteq \mathscr{T}e_0W$. Recall $e_0 = e_0u_0$ by Corollary 11.3(i) and recall that u_0 is central in $\mathscr{T}$, so

$$\begin{aligned} \mathscr{T}e_0W &= \mathscr{T}e_0u_0W \\ &\subseteq u_0W. \end{aligned}$$

From these remarks, we see that u_0W, $\mathscr{C}^*e_0W$, and $\mathscr{T}e_0W$ are all equal. Reversing the roles of C and C^* in the above argument, we see that u_0W, $\mathscr{C}e_0^*W$, and $\mathscr{T}e_0^*W$ are all equal. ∎

Now we relate u_0W to the primary module.

Proposition 14.3. *With reference to Definition* 4.1, *suppose W is a $\mathscr{T}$-module. For any irreducible $\mathscr{T}$-submodule V of W, the following are equivalent.*

(i) *$V \subseteq u_0W$.*
(ii) *V is $\mathscr{T}$-module isomorphic to the primary module.*

Proof. (i) ⇒ (ii) In view of Propositions 8.3(ii) and 8.4, we need only show that $e_0 V \neq 0$. To do this, observe that

$$\begin{aligned} V &= u_0 V \\ &\subseteq \mathscr{T} e_0 \mathscr{T} V \qquad (\text{by } (72)) \\ &\subseteq \mathscr{T} e_0 V \qquad (\text{since } V \text{ is a } \mathscr{T}\text{-module}), \end{aligned}$$

so $e_0 V \neq 0$, as desired.

(ii) ⇒ (i) Recall that u_0 acts as the identity on the primary module so $V = u_0 V \subseteq u_0 W$. ∎

With reference to Definition 4.1, suppose W is a $\mathscr{T}$-module. Since u_0 acts as the identity on $u_0 W$, we may view $u_0 W$ as a module for $\mathscr{T} u_0$. We showed in Theorem 12.3 that $\mathscr{T} u_0$ is $\mathbb{C}$-algebra isomorphic to $M_{d+1}(\mathbb{C})$, so $u_0 W$ is completely reducible. We conclude the paper by mentioning a few consequences of these ideas.

PROPOSITION 14.4. *With reference to Definition* 4.1, *suppose W is a finite dimensional $\mathscr{T}$-module. For every $\mathscr{T}$-submodule A of $u_0 W$ there exists a $\mathscr{T}$-submodule B of $u_0 W$ such that*

$$u_0 W = A + B \qquad (\text{direct sum}).$$

PROPOSITION 14.5. *With reference to Definition* 4.1, *suppose W is a finite dimensional $\mathscr{T}$-module. Then there exist a nonnegative integer m and $\mathscr{T}$-submodules $U_1, \dots, U_m$ of $u_0 W$ such that each U_i is $\mathscr{T}$-module isomorphic to the primary module and such that*

$$u_0 W = \sum_{i=1}^{m} U_i \qquad (\text{direct sum}). \tag{84}$$

PROPOSITION 14.6. *With reference to Definition* 4.1, *suppose W is a finite dimensional $\mathscr{T}$-module. Then $u_0 W = \sum U$, where the sum is over all $\mathscr{T}$-submodules U of W such that U is isomorphic to the primary module.*

Proof. This is immediate from Propositions 14.3 and 14.5. ∎

PROPOSITION 14.7. *With reference to Definition* 4.1, *suppose W is a finite dimensional $\mathscr{T}$-module. Then*

$$\dim e_0 W = \dim e_0^* W, \tag{85}$$

and this quantity is equal to the multiplicity with which the primary module appears in $u_0 W$.

Proof. We show that both sides of (85) are equal to the multiplicity with which the primary module appears in $u_0 W$. To do this, first observe that this multiplicity is given by m in (84). Now apply e_0 to both sides of (84) and use Corollary 11.3(i) to obtain

$$e_0 W = \sum_{i=1}^{m} e_0 U_i \qquad (\text{direct sum}).$$

It follows that

$$\begin{aligned}\dim e_0 W &= \sum_{i=1}^{m} \dim e_0 U_i \\ &= \sum_{i=1}^{m} 1 \qquad (\text{by Proposition 8.3(i)}) \\ &= m.\end{aligned}$$

Reversing the roles of C and C^* in the above argument, we find that $\dim e_0^* W = m$, as desired. ∎

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